Model Efek Random Dengan Interaksi

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I. Model Efek Campuran

Model : $y_{ijk} = \mu + a_i + b_j + c_{ij} + \varepsilon_{ijk}, i = 1, 2, ..., I;\ j = 1, 2, ..., J; k = 1, 2, ..., K$

Asumsi:

a. $\varepsilon_{ij} \sim N\left(0, \sigma^2\right) \Rightarrow E\left(\varepsilon_{ij}\right) = 0,\ E\left(\varepsilon_{ij}^2\right) = \text{var}\left(\varepsilon_{ij}\right) = \sigma^2$ untuk semua i, j,

b. $E\left(\varepsilon_{ij}, \varepsilon_{rs}\right) = \text{cov}\left(\varepsilon_{ij}, \varepsilon_{rs}\right) = 0$ untuk semua $i \neq r$ atau $j \neq s$,

c. $a_i \sim N\left(0, \sigma_a^2\right) \Rightarrow E\left(a_i\right) = 0,\ E\left(a_i^2\right) = \text{var}\left(a_i\right) = \sigma_a^2$ untuk semua i,

d. $E\left(a_i a_j\right) = \text{cov}\left(a_i, a_j\right) = 0$ untuk semua $i \neq j$,

e. $E\left(a_i \varepsilon_{ij}\right) = \text{cov}\left(a_i, \varepsilon_{ij}\right) = 0$ untuk semua $i \neq j$,

f. $b_j \sim N\left(0, \sigma_a^2\right) \Rightarrow E\left(b_j\right) = 0,\ E\left(b_j^2\right) = \text{var}\left(b_j\right) = \sigma_b^2$ untuk semua j,

g. $E\left(a_j a_l\right) = \text{cov}\left(a_j, a_l\right) = 0$ untuk semua $j \neq l$,

h. $E\left(b_j \varepsilon_{jl}\right) = \text{cov}\left(b_j, \varepsilon_{jl}\right) = 0$ untuk semua $j \neq l$,

i. $c_{ij} \sim N\left(0, \sigma_c^2\right) \Rightarrow E\left(c_{ij}\right) = 0,\ E\left(c_{ij}^2\right) = \text{var}\left(c_{ij}\right) = \sigma_c^2$ untuk semua i,

j. $E\left(c_{ij} c_{lj}\right) = \text{cov}\left(c_{ij}, c_{lj}\right) = 0$ untuk semua $i \neq l$,

k. $E\left(c_{ij} \varepsilon_{lj}\right) = \text{cov}\left(c_{ij}, \varepsilon_{lj}\right) = 0$ untuk semua $i \neq l$,

Akibatnya,

l. $E\left(y_{ijk}\right) = E\left(\mu + a_i + b_j + c_{ij} + \varepsilon_{ijk}\right) = \mu$,

m. $\text{var}\left(y_{ijk}\right) = \text{var}\left(\mu + a_i + b_j + c_{ij} + \varepsilon_{ijk}\right) = \sigma_a^2 + \sigma_b^2 + \sigma_c^2 + \sigma^2$untuk semua i, j, k,

n. $\text{cov}\left(y_{ijk}, y_{irk}\right) = \sigma_a^2 + \sigma_b^2 + \sigma_c^2$ untuk semua $j \neq r$,

o. $\text{cov}\left(y_{ijk}, y_{rsk}\right) = 0$ untuk semua $i \neq r$ atau $j \neq s$.

II. Estimasi Parameter

a. $\hat{\mu} = \overline{y}_{...} = \dfrac{1}{IJK}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} y_{ijk} = \dfrac{1}{IJK}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\mu + a_i + b_j + c_{ij} + \varepsilon_{ijk}\right)$

$$= \frac{1}{IJK}\left(IJK\mu + JK\sum_{i=1}^{I} a_i + IK\sum_{j=1}^{J} b_j + K\sum_{i=1}^{I}\sum_{j=1}^{J} c_{ij} \right) + \frac{1}{IJK}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\varepsilon_{ijk}$$

$$= \mu + \bar{a}_. + \bar{b}_. + \bar{c}_{..} + \bar{\varepsilon}_{...} \Rightarrow \bar{y}_{...} \sim N\left(\mu, \frac{\sigma_a^2}{I} + \frac{\sigma_b^2}{J} + \frac{\sigma_c^2}{IJ} + \frac{\sigma^2}{IJK} \right)$$

b. $\hat{a}_i = \bar{y}_{i..} - \bar{y}_{...} = \dfrac{\sum_{j=1}^{J}\sum_{k=1}^{K} y_{ijk}}{JK} - \left(\mu + \bar{a}_. + \bar{b}_. + \bar{c}_{..} + \bar{\varepsilon}_{...} \right)$

$$= \frac{\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\mu + a_i + b_j + c_{ij} + \varepsilon_{ijk} \right)}{JK} - \left(\mu + \bar{a}_. + \bar{b}_. + \bar{c}_{..} + \bar{\varepsilon}_{...} \right)$$

$$= \mu + a_i + \bar{b}_. + \bar{c}_{i.} + \bar{\varepsilon}_{i..} - \left(\mu + \bar{a}_. + \bar{b}_. + \bar{c}_{..} + \bar{\varepsilon}_{...} \right) = a_i + \bar{c}_{i.} + \bar{\varepsilon}_{i..} - \left(\bar{a}_. + \bar{c}_{..} + \bar{\varepsilon}_{...} \right)$$

$$= \left(a_i - \bar{a}_. \right) + \left(\bar{c}_{i.} - \bar{c}_{..} \right) + \left(\bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...} \right)$$

c. $\hat{\gamma}_j = \bar{y}_{.j.} - \bar{y}_{...} = \dfrac{\sum_{i=1}^{I}\sum_{k=1}^{K} y_{ijk}}{IK} - \left(\mu + \bar{a}_. + \bar{b}_. + \bar{c}_{..} + \bar{\varepsilon}_{...} \right)$

$$= \frac{\sum_{i=1}^{I}\sum_{k=1}^{K}\left(\mu + a_i + b_j + c_{ij} + \varepsilon_{ijk} \right)}{IK} - \left(\mu + \bar{a}_. + \bar{b}_. + \bar{c}_{..} + \bar{\varepsilon}_{...} \right)$$

$$= \mu + \bar{a}_. + b_j + \bar{c}_{.j} + \bar{\varepsilon}_{.j.} - \left(\mu + \bar{a}_. + \bar{b}_. + \bar{c}_{..} + \bar{\varepsilon}_{...} \right) = b_j + \bar{c}_{.j} + \bar{\varepsilon}_{.j.} - \left(\bar{b}_j + \bar{c}_{..} + \bar{\varepsilon}_{...} \right)$$

$$= \left(b_j - \bar{b}_. \right) + \left(\bar{c}_{.j} - \bar{c}_{..} \right) + \left(\bar{\varepsilon}_{.j.} - \bar{\varepsilon}_{...} \right)$$

d. $\hat{c}_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$

II. Sum of Squares dan Expected Mean of Squares

a. $SSA = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\bar{y}_{i..} - \bar{y}_{...} \right)^2 = JK\sum_{i=1}^{I}\left(\bar{y}_{i..} - \bar{y}_{...} \right)^2$

$$= JK\sum_{i=1}^{I}\left(\left(a_i - \bar{a}_. \right) + \left(\bar{c}_{i.} - \bar{c}_{..} \right) + \left(\bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...} \right) \right)^2$$

$$= JK\sum_{i=1}^{I}\left(a_i - \bar{a}_. \right)^2 + JK\sum_{i=1}^{I}\left(\bar{c}_{i.} - \bar{c}_{..} \right)^2 + JK\sum_{i=1}^{I}\left(\bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...} \right)^2$$

$$+2JK\sum_{i=1}^{I}\left(a_i - \bar{a}_. \right)\left(\bar{c}_{i.} - \bar{c}_{..} \right) + 2JK\sum_{i=1}^{I}\left(a_i - \bar{a}_. \right)\left(\bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...} \right)$$

$$+2JK\sum_{i=1}^{I}\left(\bar{c}_{i.} - \bar{c}_{..} \right)\left(\bar{\varepsilon}_{i..} - \bar{\varepsilon}_{...} \right)$$

$$= JK\left(I-1\right)\sigma_a^2 + \frac{JK\left(I-1\right)\sigma_c^2}{J} + \frac{JK\left(I-1\right)\sigma^2}{JK}$$

$$+2JK\left(I-1\right)\text{cov}(a_i,\bar{c}_{i.}) + \frac{2JK(I-1)\,\text{cov}(a_i,\bar{\varepsilon}_{i..})}{J} + \frac{2JK(I-1)\,\text{cov}(\bar{c}_{i.},\bar{\varepsilon}_{i..})}{JK}$$

$$= JK\left(I-1\right)\sigma_a^2 + K\left(I-1\right)\sigma_c^2 + \left(I-1\right)\sigma^2$$

$$E(SSA/\left(I-1\right)) = \frac{JK\left(I-1\right)\sigma_a^2 + K\left(I-1\right)\sigma_c^2 + \left(I-1\right)\sigma^2}{\left(I-1\right)} = JK\sigma_a^2 + K\sigma_c^2 + \sigma^2$$

Dengan demikian,

$$E(MSA) = JK\sigma_a^2 + K\sigma_c^2 + \sigma^2$$

b. $SSB = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\bar{y}_{.j.} - \bar{y}_{...}\right)^2 = IK\sum_{j=1}^{J}\left(\bar{y}_{.j.} - \bar{y}_{...}\right)^2$

$$= IK\sum_{j=1}^{J}\left(\left(b_j - \bar{b}_{.}\right) + \left(\bar{c}_{.j} - \bar{c}_{..}\right) + \left(\bar{\varepsilon}_{.j.} - \bar{\varepsilon}_{...}\right)\right)^2$$

$$= IK\sum_{j=1}^{J}\left(b_j - \bar{b}_{.}\right)^2 + IK\sum_{j=1}^{J}\left(\bar{c}_{.j} - \bar{c}_{..}\right)^2 + IK\sum_{j=1}^{J}\left(\bar{\varepsilon}_{.j.} - \bar{\varepsilon}_{...}\right)^2$$

$$+2JK\sum_{i=1}^{I}\left(b_j - \bar{b}_{.}\right)\left(\bar{c}_{.j} - \bar{c}_{..}\right) + 2JK\sum_{i=1}^{I}\left(b_j - \bar{b}_{.}\right)\left(\bar{\varepsilon}_{.j.} - \bar{\varepsilon}_{...}\right)$$

$$+2JK\sum_{i=1}^{I}\left(\bar{c}_{.j} - \bar{c}_{..}\right)\left(\bar{\varepsilon}_{.j.} - \bar{\varepsilon}_{...}\right)$$

$$= IK\sum_{j=1}^{J}\left(b_j - \bar{b}_{.}\right)^2 + IK\sum_{j=1}^{J}\left(\bar{c}_{.j} - \bar{c}_{..}\right)^2 + IK\sum_{j=1}^{J}\left(\bar{\varepsilon}_{.j.} - \bar{\varepsilon}_{...}\right)^2$$

$$= IK\left(J-1\right)\sigma_b^2 + \frac{IK\left(J-1\right)\sigma_c^2}{I} + \frac{IK\left(J-1\right)\sigma^2}{IK}$$

$$= IK\left(J-1\right)\sigma_b^2 + K\left(J-1\right)\sigma_c^2 + \left(J-1\right)\sigma^2$$

$$E(SSB/\left(J-1\right)) = \frac{IK\left(J-1\right)\sigma_b^2 + K\left(J-1\right)\sigma_c^2 + \left(J-1\right)\sigma^2}{\left(J-1\right)} = IK\sigma_b^2 + K\sigma_c^2 + \sigma^2$$

Dengan demikian,

$$E(MSB) = IK\sigma_b^2 + K\sigma_c^2 + \sigma^2$$

c. $SSE = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(y_{ijk} - \bar{y}_{...}\right)^2 = IJ(K-1)\sigma^2$

d. $SSAB = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\hat{c}_2^2 = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}\right)^2$

$$\bar{y}_{ij.} = \frac{\sum_{k=1}^{K} y_{ijk}}{K} = \mu + a_i + b_j + c_{ij} + \bar{\varepsilon}_{ij.} \Rightarrow \bar{y}_{ij.} \sim N\left(\mu, \sigma_a^2 + \sigma_b^2 + \sigma_c^2 + \frac{\sigma^2}{K}\right),$$

$$\overline{y}_{i..} = \frac{\sum_{j=1}^{J}\sum_{k=1}^{K} y_{ijk}}{JK} = \mu + a_i + \overline{b}_{.} + \overline{c}_{i.} + \overline{\varepsilon}_{i..} \Rightarrow \overline{y}_{ij.} \sim N\left(\mu, \sigma_a^2 + \frac{\sigma_c^2}{J} + \frac{\sigma_c^2}{J} + \frac{\sigma^2}{JK}\right),$$

$$\overline{y}_{.j.} = \frac{\sum_{i=1}^{I}\sum_{k=1}^{K} y_{ijk}}{IK} = \mu + \overline{a}_{.} + b_j + \overline{c}_{.j} + \overline{\varepsilon}_{.j.} \Rightarrow \overline{y}_{ij.} \sim N\left(\mu, \frac{\sigma_a^2}{I} + \sigma_b^2 + \frac{\sigma_c^2}{I} + \frac{\sigma^2}{IK}\right),$$

$$E\left[MSAB\right] = E\left[\frac{SSAB}{(I-1)(J-1)}\right] = E\left[\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\overline{y}_{ij.} - \overline{y}_{i..} - \overline{y}_{.j.} + \overline{y}_{...}\right)^2}{(I-1)(J-1)}\right],$$

$$= E\left[\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(c_{ij} - \overline{c}_{i.} - \overline{c}_{.j} + \overline{c}_{..} + \overline{\varepsilon}_{ij.} - \overline{\varepsilon}_{ij.} - \overline{\varepsilon}_{ij.} + \overline{\varepsilon}_{ij.}\right)^2}{(I-1)(J-1)}\right]$$

$$= E\left[\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(c_{ij} - \overline{c}_{i.} - \overline{c}_{.j} + \overline{c}_{..}\right)^2 + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(+\overline{\varepsilon}_{ij.} - \overline{\varepsilon}_{ij.} - \overline{\varepsilon}_{ij.} + \overline{\varepsilon}_{ij.}\right)^2}{(I-1)(J-1)}\right],$$

$$= \left[\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} E\left(c_{ij} - \overline{c}_{i.} - \overline{c}_{.j} + \overline{c}_{..}\right)^2 + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} E\left(+\overline{\varepsilon}_{ij.} - \overline{\varepsilon}_{ij.} - \overline{\varepsilon}_{ij.} + \overline{\varepsilon}_{ij.}\right)^2}{(I-1)(J-1)}\right]$$

$$= \left[\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} \frac{(I-1)(J-1)}{IJ}\sigma_c^2 + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} \frac{(I-1)(J-1)}{IJK}\sigma^2}{(I-1)(J-1)}\right] = K\sigma_c^2 + \sigma^2$$

III. Tabel Anova

Sumber variasi	Sum of Squares	Derajat Bebas	Expected Mean of Squares
Faktor-1	$JK\sum_{i=1}^{I}\left(\bar{y}_{i..}-\bar{y}_{...}\right)^2$	*I-1*	$JK\sigma_a^2+K\sigma_c^2+\sigma^2$
Faktor-2	$IK\sum_{j=1}^{J}\left(\bar{y}_{.j.}-\bar{y}_{...}\right)^2$	*J-1*	$IK\sigma_b^2+K\sigma_c^2+\sigma^2$
Interaksi	$K\sum_{i=1}^{I}\sum_{k=1}^{K}\left(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...}\right)^2$	(I-1)(J-1)	$K\sigma_c^2+\sigma^2$
Error	$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(y_{ijk}-\bar{y}_{...}\right)^2$	$IJ(K$-1)	σ^2
Total	$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}y_{ijk}^2$	IJK-1	

IV. Uji Hipotesis

a. $H_0:\sigma_a^2=0$ v.s $\sigma_a^2>0$

Dengan memperhatikan Tabel ANOVA pada romawi III, pada saat H_0 benar maka EMS (*Expected Mean Squares*) dari Faktor-1 sama dengan EMS faktor interaksi. Dengan demikian, tolak Ho jika:

$$F_{hit}=\frac{MSA}{MSAB}>F_{\alpha;(I-1),(I-1)(J-1)}$$

b. $H_0:\sigma_b^2=0$ v.s $\sigma_b^2>0$

Dengan memperhatikan Tabel ANOVA pada romawi III, pada saat H_0 benar maka EMS (*Expected Mean Squares*) dari Faktor-2 sama dengan EMS faktor interaksi. Dengan demikian, tolak Ho jika:

$$F_{hit}=\frac{MSB}{MSAB}>F_{\alpha;(J-1),(I-1)(J-1)}$$

b. $H_0:\sigma_c^2=0$ v.s $\sigma_c^2>0$

Dengan memperhatikan Tabel ANOVA pada romawi III, pada saat H_0 benar maka EMS (*Expected Mean Squares*) dari faktor interaksi sama dengan EMS Error. Dengan demikian, tolak Ho jika:

$$F_{hit}=\frac{MSAB}{MSE}>F_{\alpha;(I-1)(J-1),IJ(K-1)}$$